SECTION 15.4: THE CHAIN RULE

RECALL: $D_x[f(g(x))] = f'(g(x))g'(x)$. Said differently, if we let $y = f(g(x))$ and $u = g(x)$, then:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Said differently:

rate of change of y with respect to x = (rate of change of y with respect to u)·(rate of change of u with respect to x)

Here we may think of u as an 'intermediate variable' here between y and x:

$$\begin{tikzcd} y \arrow[d] \\ u \arrow[d] \\ x \end{tikzcd}$$

We may generalize this to functions of more than one variable as follows:

THEOREM: Suppose $z = f(x, y)$ is a differentiable function of x and y where both x and y are differentiable functions of t. Then z is a differentiable function of t and, furthermore:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \qquad \Longleftrightarrow \qquad z'(t) = f_x(x, y)\, x'(t) + f_y(x, y)\, y'(t)$$

Here, we may view the 'tree of dependence' as with x and y as 'intermediate variables.'

$$\begin{tikzcd} & z \arrow[dl] \arrow[dr] & \\ x \arrow[d] & & y \arrow[d] \\ t & & t \end{tikzcd}$$

EXAMPLE 1: Translate the chain rule into a sentence about rates of change.

EXAMPLE 2: Find $\dfrac{dz}{dt}$ in terms of t where $z = x^2 y$, $x = \cos(t)$, and $y = \sin(t)$ in two ways:

1. using the multivariable chain rule.

Ans: $\dfrac{dz}{dt} = -2\cos(t)\sin^2(t) + \cos^3(t)$

2. by first substituting $x = \cos(t)$, and $y = \sin(t)$ into the formula for z and taking the derivative.

Ans: $\dfrac{dz}{dt} = -2\cos(t)\sin^2(t) + \cos^3(t)$ ✓

EXAMPLE 3: For each of the functions below, assume $x = f(t)$ and $y = g(t)$ are differentiable functions of t. Use the multivariable chain rule to find a formula for $F'(t)$ in terms of $f(t)$, $f'(t)$, $g(t)$ and $g'(t)$.

1. $F(x, y) = xy$

Ans: $F'(t) = f'(t)g(t) + f(t)g'(t)$

2. $F(x, y) = \dfrac{x}{y}$

Ans: $F'(t) = \dfrac{f'(t)}{g(t)} - \dfrac{f(t)g'(t)}{g(t)^2} = \dfrac{g(t)f'(t) - f(t)g'(t)}{g(t)^2}$

Do your answers look familiar? (They should!)

EXAMPLE 4: The temperature T in degrees Fahrenheit on a plate at location (x, y) is given by:

$$T(x, y) = \frac{100 \ln(y)}{x} \qquad x, y > 0.$$

Suppose an object is traversing the plate with position $\vec{r}(t) = \langle x(t), y(t) \rangle$.

1. Use the multivariable chain rule to find an expression for $T'(t)$ in terms of $x(t)$, $y(t)$, $x'(t)$ and $y'(t)$.

Ans: $T'(t) = -\frac{100 \ln(y(t))}{x(t)^2}\, x'(t) + \frac{100}{x(t)y(t)}\, y'(t)$

2. Use your formula for $T'(t)$ to simplify $T'(t)$ for $\vec{r}(t) = \left\langle t, e^{2t} \right\rangle$. Interpret your answer.

Ans: $T'(t) = 0$. On $\vec{r}(t) = \left\langle t, e^{2t} \right\rangle$, the temperature is constant! (Indeed, $T(t) = 200$.)

EXAMPLE 5: Suppose $\vec{r}(t) = \langle x(t), y(t) \rangle$ traces out the level curve of a differentiable function $f(x, y) = c$.

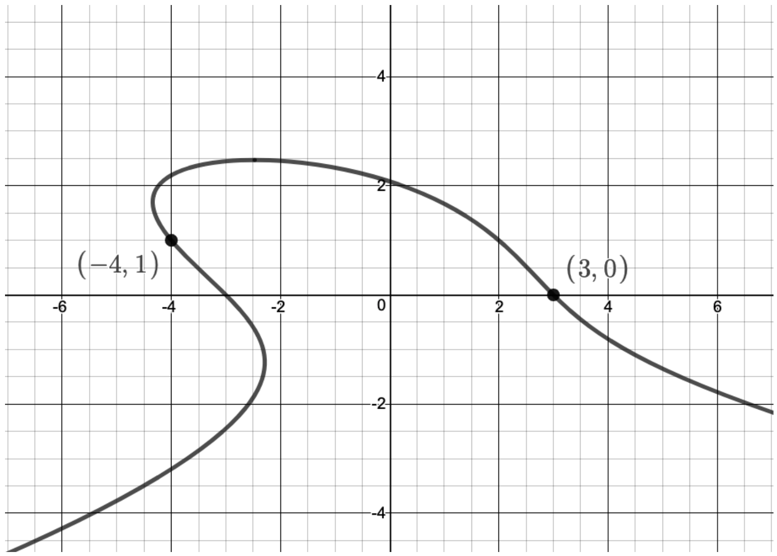
1. Differentiate both sides of $f(x, y) = c$ with respect to t to prove: $f_x(x, y)x'(t) + f_y(x, y)y'(t) = 0$.

2. Let $\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle$. Use the previous result to show $\nabla f(x, y) \perp \vec{r}\,'(t)$.

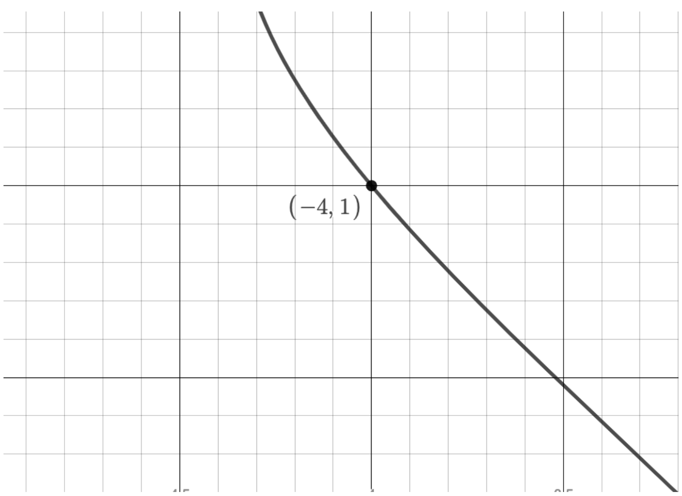
IMPLICIT DIFFERENTIATION REVISITED

RECALL: Some functional relationships exist only locally and are described best **implicitly** using an equation.

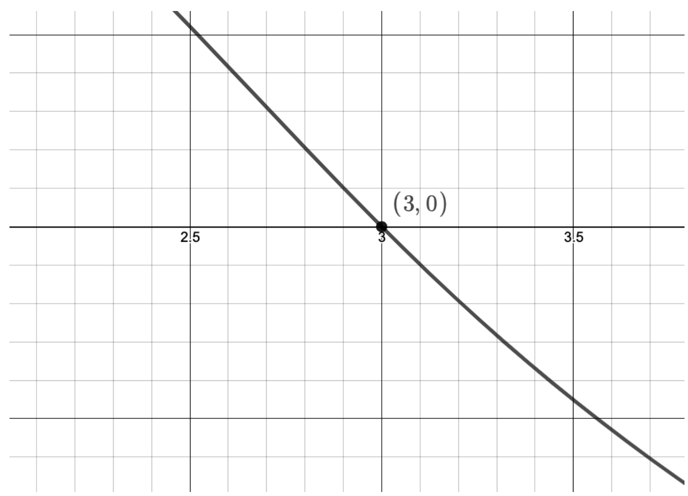
Consider the graph of $x^2 + 2xy + y^3 = 9$ below.



Even though the graph fails the vertical line test, **locally** this curve represents a differentiable function of x if we zoom in near (most every) point. For instance, if we 'zoom in' near the point $(-4, 1)$ and again near the point $(3, 0)$, there is a window in which the graph **does** pass the vertical line test. Hence, **locally**, y is a function of x.



'near' $(-4, 1)$



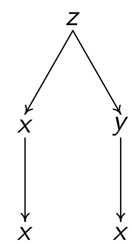
'near' $(3, 0)$

In Calculus 1, we learned a technique called **implicit differentiation** that allowed us to find an expression for $\dfrac{dy}{dx}$ in terms of x and y without having to actually solve $y = f(x)$. In Calculus 3, we revisit this notion.

THEOREM: If $z = f(x, y)$ is a differentiable function of x and y and the equation $f(x, y) = c$ implicitly describes y as a differentiable function of x, then

$$\frac{dy}{dx} = -\frac{f_x(x, y)}{f_y(x, y)}$$

PROOF: Consider the tree diagram:



The chain rule gives: $\frac{dz}{dx} = f_x(x, y)\frac{dx}{dx} + f_y(x, y)\frac{dy}{dx} = f_x(x, y) + f_y(x, y)\frac{dy}{dx}$.

Differentiating both sides of $f(x, y) = c$ with respect to x gives: $\frac{d}{dx}[f(x, y)] = \frac{d}{dx}[c]$ so $f_x(x, y) + f_y(x, y)\frac{dy}{dx} = 0$.

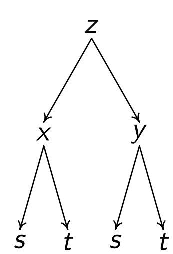
Solving $f_x(x, y) + f_y(x, y)\frac{dy}{dx} = 0$ for $\frac{dy}{dx}$ gives $\frac{dy}{dx} = -\frac{f_x(x, y)}{f_y(x, y)}$.

EXAMPLE 6: Find an expression for $\frac{dy}{dx}$ in in terms of x and y if $x^2 + 2xy + y^3 = 9$.

Ans: $\frac{dy}{dx} = -\frac{2x + 2y}{2x + 3y^2}$

EXTENSIONS TO MORE VARIABLES

Suppose $z = f(x, y)$ is a differentiable function of x and y where x and y are each differentiable functions of s and t. Then, ultimately, z is a function of s and t as seen in the tree below.



Use this tree to find expressions for:

1. $\frac{\partial z}{\partial s} =$

2. $\frac{\partial z}{\partial t} =$

EXAMPLE 7: Let $z = f(x, y) = 4 - 3x^2 - 3y^2$ where $x = r\cos(\theta)$ and $y = r\sin(\theta)$.

Use the chain rule to find and simplify formulas for

1. $\dfrac{\partial z}{\partial r} =$

Ans: $\dfrac{\partial z}{\partial r} = -6r$

2. $\dfrac{\partial z}{\partial \theta} =$

Ans: $\dfrac{\partial z}{\partial \theta} = 0$

3. Rewrite $z = f(x, y)$ in terms of r and θ to explain your answers!

Ans: $z = 4 - 3r^2$

EXAMPLE 8: Suppose $f(x, y)$ has continuous partial derivatives and that $x = r\cos(\theta)$ and $y = r\sin(\theta)$.

1. Show:

$$f_\theta(x, y) = xf_y(x, y) - yf_x(x, y)$$

2. **BONUS:** Use your result above to show:

$$f_{\theta\theta}(x, y) = y^2 f_{xx}(x, y) - 2xyf_{xy}(x, y) + x^2 f_{yy}(x, y) - xf_x(x, y) - yf_y(x, y)$$

EXAMPLE 9: Suppose $w = F(x, y, z)$ is a differentiable function of x, y, and z where

$$x = \rho\sin(\phi)\cos(\theta), \quad y = \rho\sin(\phi)\sin(\theta), \quad \text{and} \quad z = \rho\cos(\phi)$$

1. Draw a tree of dependence for this situation.

2. Find and simplify expressions for the following in terms of ρ, ϕ, and/or θ:

$$\frac{\partial w}{\partial \rho} =$$

Ans: $\dfrac{\partial w}{\partial \rho} = F_x(x, y, z)\sin(\phi)\cos(\theta) + F_y(x, y, z)\sin(\phi)\sin(\theta) + F_z(x, y, z)\cos(\phi)$

$$\frac{\partial w}{\partial \phi} =$$

Ans: $\dfrac{\partial w}{\partial \phi} = F_x(x, y, z)\rho\cos(\phi)\cos(\theta) + F_y(x, y, z)\rho\cos(\phi)\sin(\theta) - F_z(x, y, z)\rho\sin(\phi)$

$$\frac{\partial w}{\partial \theta} =$$

Ans: $\dfrac{\partial w}{\partial \theta} = -F_x(x, y, z)\rho\sin(\phi)\sin(\theta) + F_y(x, y, z)\rho\sin(\phi)\cos(\theta)$

EXAMPLE 10: Suppose $w = F(x, y, z)$ where $x = x(s, t)$, $y = y(t)$, and $z = z(s)$.

1. Draw a tree of dependence for this situation.

2. Assuming w, x, y, and z are differentiable, find an expression for $\dfrac{\partial w}{\partial s}$.

Ans: $\dfrac{\partial w}{\partial s} = F_x(x, y, z)\dfrac{\partial x}{\partial s} + F_z(x, y, z)\dfrac{dz}{ds}$

HOMEWORK: Section 15.4: 9 - 73 every other odd